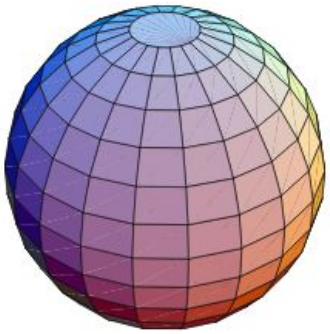
Lecture Notes on Sphere

Prasad T.Bathe
Assistant Professor in Mathematics,
Rayat Shikshan Sanstha's
Mahatma Phule Mahavidyalaya, Pimpri, Pune-411017

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1 Definition

A *Sphere* is the locus of a point which remains at a constant distance from a fixed point.The constant distance is called the radius and the fixed point is called as centre of the sphere.



2 Equation of a Sphere

2.1 General Equation of Sphere

Any second degree equation in which coefficient of x^2, y^2, z^2 are all non-zero and equal , and product terms xy, yz, zx are absent , that is , equation of the type

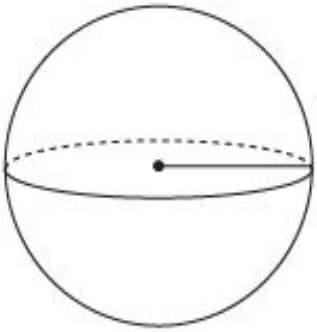
$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

where u, v, w, d are constants and $u^2 + v^2 + w^2 - d \geq 0$, represents a sphere with centre $(-u, -v, -w)$ and radius $\sqrt{u^2 + v^2 + w^2 - d}$

2.2 Centre Radius Form

$$x^2 + y^2 + z^2 - 2ax - 2by - 2cz + \left(a^2 + b^2 + c^2 - r^2\right) = 0$$

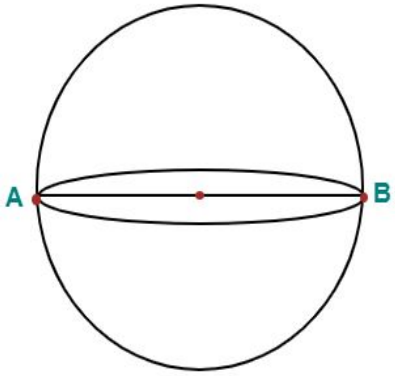
is known as equation of sphere with centre at a point (a, b, c) and radius r



2.3 Diameter Form of a Equation of Sphere

The Equation of a Sphere with segment AB as a diameter , where $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ is given by

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z_2) = 0$$



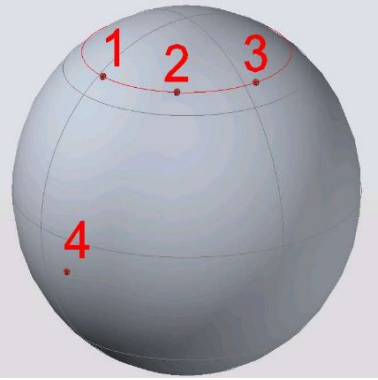
2.4 Equation of the Sphere through Four given points

The equation of the sphere passing through four given points

$$A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3), D(x_4, y_4, z_4)$$

is

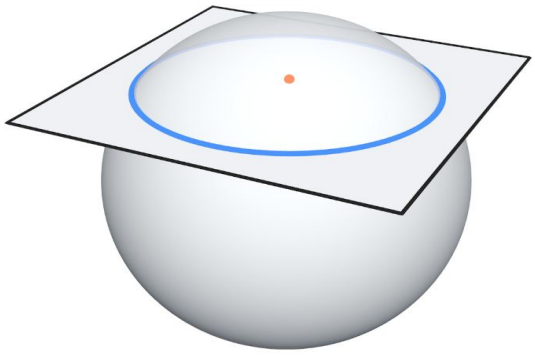
$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$



3 Equations of a Circle

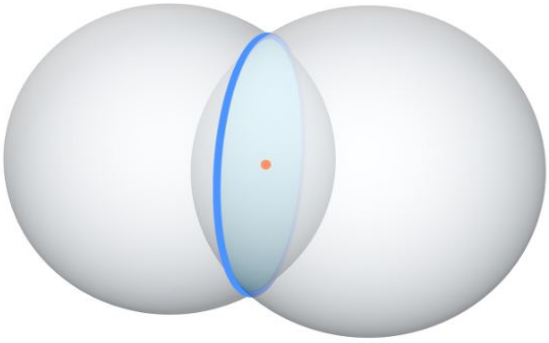
3.1 Plane Section of a Sphere

The set of points common to both sphere and plane is called a plane section of a sphere. The equations of the sphere and the plane taken together represent the plane section.



3.2 Intersection of two Spheres

The curve of intersection of two spheres is a circle.



The co-ordinates of points common to any two spheres

$$S_1 \equiv x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$$
$$S_2 \equiv x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$$

satisfy both these equations and , therefore , they also satisfy the equation

$$S_1 - S_2 \equiv 2x\,(u_1 - u_2) + 2y\,(v_1 - v_2) + 2z\,(w_1 - w_2) + (d_1 - d_2) = 0$$

which is the first degree equation in x, y, z represents a plane and the intersection of sphere and plane represents a circle. Thus two equations

$$S_1 \equiv \qquad x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$$
$$S_2 \equiv \qquad x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$$

taken together represents a circle.

3.3 Sphere through a given Circle

The equation of sphere which passes through the circle with equations

$$S \equiv \qquad x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad (1)$$
$$U \equiv \qquad lx + my + nz - p = 0 = 0 \qquad (2)$$

is given by

$$S + \lambda U = 0$$

3.4 Intersection of a Sphere and a line

A line intersects a sphere in at most two points , that is , every straight line meets a sphere in two points which may be real , imaginary or coincident.
Let the equation of Sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad (3)$$

and the equations of a line be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r(say) \tag{4}$$

The point $(lr+\alpha, mr+\beta, nr+\gamma)$ which lies on the given line (4) for all values of r, will also lie on the given sphere (3), if r satisfies the equation (3) , that is

$$(lr+\alpha)^2+(mr+\beta)^2+(nr+\gamma)^2+2u\,(lr+\alpha)+2v\,(mr+\beta)+2w\,(nr+\gamma)+d=0 \tag{5}$$

$$r^2\left(l^2+m^2+n^2\right)+2r\left[l(\alpha+u)+m(\beta+v)+n(\gamma+w)\right]+\left(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d\right)=0 \tag{6}$$

The equation (6) is quadratic in r and it has two roots say r_1 and r_2 .

- If r_1 and r_2 are real and distinct , then there two common points.
- If $r_1=r_2$ then line touches the sphere.
- If r_1 and r_2 are imaginary , then line does not intersect the sphere.

3.5 Equation of Tangent Plane

The equation of the tangent plane at any point $P(x_1, y_1, z_1)$ of the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

is

$$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$$

